

Pairs of Additive Sextic Forms

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Abstract

A special case of a conjecture attributed to Artin states that any system of two homogeneous diagonal forms of degree k with integer coefficients should have nontrivial zeros over any p -adic field \mathbb{Q}_p provided only that the number of variables is at least $2k^2 + 1$. In this article, we prove that the conjecture is true when $k = 6$.

Keywords: p -adic solubility, Artin's conjecture, sextic forms

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1. Introduction

A special case of a conjecture commonly attributed to Emil Artin states that any system of equations

$$\begin{aligned} a_1x_1^k + a_2x_2^k + \cdots + a_sx_s^k &= 0 \\ b_1x_1^k + b_2x_2^k + \cdots + b_sx_s^k &= 0, \end{aligned}$$

where the a_i and b_i are integers, should have nontrivial solutions in every p -adic field \mathbb{Q}_p provided only that $s \geq 2k^2 + 1$, where “nontrivial” simply means that at least one of the variables should be different from zero. In many cases, this is known to be true. As part of their pioneering work on this and similar problems, Davenport & Lewis [1] showed that the conjecture

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is true when k is an odd integer, but only that $7k^3$ variables suffice when k is even. Some years later, Brüdern & Godinho [2] proved that the conjecture is true for most even exponents, leaving only the cases where $k = p^\tau(p - 1)$ with $\tau \geq 1$ and where $k = 3 \cdot 2^\tau$ as possible exceptions. Even in these exceptional cases, it is known that the system must always have nontrivial q -adic solutions for a prime q unless either $q = p$ in the first case or $q = 2$ in the second case. We note that $k = 6$ is the only degree to fall into both classes of exceptions. It is our goal in this paper to prove that the conjecture holds when $k = 6$. That is, we will prove the following theorem.

Theorem 1. *Suppose that $a_1, \dots, a_s, b_1, \dots, b_s \in \mathbb{Z}$. If $s \geq 73$, then the system of equations*

$$\begin{aligned} a_1x_1^6 + a_2x_2^6 + \dots + a_sx_s^6 &= 0 \\ b_1x_1^6 + b_2x_2^6 + \dots + b_sx_s^6 &= 0 \end{aligned} \tag{1}$$

has nontrivial solutions in each p -adic field \mathbb{Q}_p .

According to the aforementioned results of Brüdern & Godinho, the theorem is true for all primes $p > 3$, so our proof focuses on the primes 2 and 3. Our goal is to find a nonsingular solution of the system (1) modulo a suitable power of p , and then lift this to a p -adic solution using Hensel's Lemma. Our primary technique is the method of contractions, which essentially involves building up solutions of congruences one power of p at a time. We improve on previous work by combining this technique with both the colored variables technology developed by Brüdern and Godinho and the theory of zero-sum sequences in groups. We also improve on previous work in that frequently, in order to make our contractions, we consider the coefficients of variables modulo p^2 (or even modulo p^3) instead of only modulo p , which has been typical previously.

In Section 2 of this article, we present various preliminary lemmas which apply for all (or almost all) values of p . Then Section 3 will deal with 2-adic solubility, and Section 4 will treat the 3-adic case. For both $p = 2$ and $p = 3$ separately, we prove various propositions which will only be used for that particular value of p , and these will be included in Section 3 and Section 4 respectively, instead of in the more general Section 2.

2. Preliminaries

In this section, we record a few definitions and preliminary results that we need in our proof. The lemmata in this section will be applied to both the primes 2 and 3 in the following sections. Preliminaries that only apply to one of these primes will be presented in the section devoted to that prime. Our first lemma is a consequence of the two main theorems of [2]. This lemma implies that we only need to consider the primes 2 and 3 in our investigations.

Lemma 1 (Brüderer & Godinho). *Fix a prime number p and suppose that the coefficients $a_i, b_i, 1 \leq i \leq s$ in the equation (1) are all ordinary integers. If $s \geq 2k^2 + 1$ and neither of the exceptional conditions below occur, then the equation (1) is guaranteed to have nontrivial solutions in \mathbb{Q}_p :*

- $k = p^\tau(p - 1)$ for some $\tau \geq 1$
- $p = 2$ and $k = 3 \cdot 2^\tau$ for some $\tau \geq 1$.

Our next lemma is a combination of several results in [1], specialized to degree 6. This allows us to assume that our system of equations has certain special properties. In this lemma, the phrase “we may assume” means that if all systems of equations with these properties have nontrivial p -adic solutions, then all systems without these properties must have nontrivial solutions as well. Therefore, we are free to make these assumptions about the system, and do so from this point onward unless otherwise specified. A system satisfying the properties of this lemma will be said to be *p -normalized*.

Lemma 2. *Consider a system of equations*

$$\begin{aligned} f &= a_1x_1^6 + \cdots + a_sx_s^6 = 0 \\ g &= b_1x_1^6 + \cdots + b_sx_s^6 = 0, \end{aligned} \tag{2}$$

where all of the coefficients are integers, and fix a prime number p . We may rewrite the polynomials f and g as

$$f = \sum_{j=0}^5 p^j f_j, \quad g = \sum_{j=0}^5 p^j g_j,$$

where for each j , the functions f_j and g_j are additive forms with integer coefficients, and for each variable involved in the pair f_j, g_j , the coefficient

of this variable in at least one of the forms is not divisible by p . For each j , let m_j represent the total number of variables involved in the pair f_j, g_j , and let q_j represent the minimal number of variables in any nontrivial linear combination of these forms. Then we may assume that for $0 \leq j \leq 5$, we have

$$m_0 + \dots + m_j \geq \frac{(j+1)s}{6} \quad \text{and} \quad m_0 + \dots + m_{j-1} + q_j \geq \frac{(2j+1)s}{12}.$$

Moreover, we may assume that g_0 contains exactly q_0 variables with coefficients not divisible by p , and that if t represents the number of variables in g_0 with coefficients divisible by p^2 , then we have

$$m_0 + u(g_1) - s/k \geq t \geq (m_0 - q_0)/p,$$

where $u(g_1)$ represents the number of variables in g_1 whose coefficients are nonzero modulo p .

As mentioned in the introduction, our strategy is to solve our system modulo a power of p and then use Hensel's Lemma to obtain p -adic solutions. The following version of Hensel's Lemma is standard for this.

Lemma 3. *Consider the system (1). Fix a prime p , and write $k = p^\tau k_0$, where $(p, k_0) = 1$. Define the number $\gamma = \gamma(k, p)$ by*

$$\gamma = \begin{cases} \tau + 2 & \text{if } p = 2 \text{ and } \tau > 0 \\ \tau + 1 & \text{otherwise.} \end{cases}$$

Suppose that we can find a solution to the system modulo p^γ such that there exist indices i, j such that

$$(a_i b_j - a_j b_i) x_i x_j \not\equiv 0 \pmod{p}. \quad (3)$$

Then this solution of congruences lifts to a p -adic solution of (1).

Our primary method in the proofs is the technique of *contractions* developed by Davenport & Lewis. We now briefly sketch the ideas and terminology involved. We say that a variable y in our system is at *level* j if it is a variable in the pair f_j, g_j in Lemma 2. Suppose that we have variables y_1, \dots, y_n at level l , and that $\mathbf{y} = \xi$ is a solution of the system

$$\sum_{i=1}^n a_i y_i^6 \equiv \sum_{i=1}^n b_i y_i^6 \equiv 0 \pmod{p^l}$$

in which $\xi_i \not\equiv 0 \pmod{p}$ for each i . With this solution, we may define a new variable Y by setting $y_1 = \xi_1 Y, \dots, y_n = \xi_n Y$. If Y is a variable at level m , we call this a *contraction of variables at level l to a variable at level m* . In this definition, if $l = 0$ and in the solution of congruences modulo p , there are indices i, j such that $(a_i b_j - a_j b_i) \xi_i \xi_j \not\equiv 0 \pmod{p}$, then we call Y a *primary* variable at level m . Similarly, if Y was obtained by (perhaps several) contractions of variables, and one of the variables involved in the contractions is a primary variable, then Y is also said to be primary. If Y is not a primary variable, then we call Y a *secondary* variable. Note that if we can create a primary variable at level γ or higher, then by setting this variable equal to 1, we obtain a solution of (1) modulo p^γ which satisfies the conditions of Lemma 3. Hence, if we can create a primary variable at level at least γ , then we know that the system (1) has a nontrivial p -adic solution.

Let us note here one subtle point in our argument. If we have some variables at level l and are able to contract them to a primary variable, then the worst-case scenario is always that the new variable is at level $l + 1$. This is because our goal is to construct primary variables at successively higher levels. If our contraction yields a primary variable at a level higher than $l + 1$, then that variable is already there without us having to construct it. Thus, when we say that we can construct a primary variable at level $l + 1$, we really mean that we can construct it at level $l + 1$ or higher, and it is understood that if this variable is actually at a higher level, then the proof of the theorem becomes simpler. Secondary variables, however, are needed at a particular level in order to guarantee that they can be used to construct primary variables. Thus, when we say that we construct a secondary variable at level $l + 1$, we must take care to ensure that this variable is at level exactly $l + 1$.

In our proofs of both 2-adic and 3-adic solubility, we make great use of the colored variables technology developed by Brüdern & Godinho, and so we record here some of the basic ideas about colored variables. As before, suppose that the prime p is fixed, and that x_i is a variable in (1) at level l . Then both of the coefficients a_i and b_i are divisible by p^l , and at least one coefficient is not divisible by p^{l+1} . By the *color* of the variable x_i , we mean the ratio a_i/b_i , considered as an element modulo p , unless $b_i = 0$ when we

say that x_i has color 0. Thus there are $p + 1$ possible colors:

$$\mathbf{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \dots, \mathbf{e}_p = \begin{pmatrix} p \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For convenience later, we define the numbers $i_j, 1 \leq j \leq p$ to be the number of variables at level 0 having color \mathbf{e}_j . If we have several variables at some level, then we use the term *palette* to refer to the set of colors (including multiplicity) of these variables.

At times, we will be interested in the vector of the values of the coefficients of a variable instead of just the variable's color. In this case, we will use curly brackets and write this vector as $\begin{Bmatrix} a \\ b \end{Bmatrix}$. Suppose that we have a sequence of variables whose coefficient vectors are

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}, \dots, \begin{Bmatrix} a_n \\ b_n \end{Bmatrix},$$

and such that

$$\sum_{i=0}^n \begin{Bmatrix} a_i \\ b_i \end{Bmatrix} \equiv \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \pmod{p^l}.$$

Then we say that this sequence is a *zero-sum sequence modulo p^l* . If a subset of these coefficient vectors sums to zero, then we call this subset a *zero-sum subsequence*. Note that if we have a zero-sum sequence, then we can contract these variables to a higher level. Note also that a variable Y at level l is primary if and only if when we trace back the variables used in contractions to create Y , we find that we have used two variables of different colors at level 0. Usually, when we need the coefficient vector of a variable at level l , we will only consider the vector of coefficients in f_l and g_l , suppressing the implied factor of p^l .

Our next lemma, due to Davenport & Lewis [1], gives us a lower bound on the number of primary variables at level 1 which we are able to construct through contractions.

Lemma 4. *Let $\delta = (k, p - 1)$. If π_1 represents the number of primary variables at level 1 which we can create by contracting variables at level 0, then we have*

$$\pi_1 \geq \min \left\{ \left\lfloor \frac{m_0}{2\delta + 1} \right\rfloor, \left\lfloor \frac{q_0}{\delta + 1} \right\rfloor \right\}.$$

Moreover, in each of these contractions, we use at most $\delta + 1$ variables of the majority color at level 0.

The following two lemmas give some more general information about when we can contract variables to higher levels. The first of these is due to Olson (see [3] and [4]), and the second is due to Godinho & de Souza Neto [5].

Lemma 5 (Olson). *Let p be a fixed prime, and suppose that S is a sequence of variables at level l , having length n . Then*

1. *if $n \geq 2p - 1$, then S has a zero-sum subsequence modulo p^{l+1} ;*
2. *if $n \geq 3p - 2$, then S has a zero-sum subsequence modulo p^{l+1} having length at most p .*

Lemma 6 (Godinho & de Souza Neto). *Let p be a fixed prime, and suppose that S is a sequence of variables at level l .*

1. *If we have $i_j(S) \geq p$ for some j , then for any element v of the sequence, we can find a zero-sum subsequence modulo p^{l+1} of S which includes the element v .*
2. *If $p = 3$ or $p = 5$, and we have $i_j(S) \geq 2p - 1$ for some j , then we can find a zero-sum subsequence modulo p^{l+1} which is **not** a zero-sum sequence modulo p^{l+2} , and which has length at most p .*

At times in our proof, we will make some contractions and then be interested in the remaining numbers of variables at a given level. In these situations, we use notation with primes to denote the new numbers of variables. For example, if we contract some variables from level 0 to level 1, then we will denote the number of remaining variables at level 0 by m'_0 and the new number of variables with color 0 as i'_0 . We also note that all of our theorems about contractions (and in particular Lemma 4) still apply when the variables in the lemma are replaced by their corresponding primed variables.

Our final lemma in this section is an extension of a result due to Bovey [6], which gives us a condition under which we can guarantee that we can solve congruences modulo powers of primes. Although we only use this lemma for the prime $p = 3$, we include it in this section since the result applies to any prime. Although Bovey only states this result for $p = 2$, his proof extends to any p with no extra work, and so we will not include a proof here.

Lemma 7. *Let $n \in \mathbb{Z}^+$, and suppose that for $i = 0, \dots, n$, we have $F_i = \sum_{j=1}^{v_i} a_{ij}x_{ij}$ with all $a_{ij} \not\equiv 0 \pmod{p}$ and with $\sum_{i=0}^{k-1} v_i \geq p^k$ for each $k = 1, \dots, n$. Then for any positive integer $N > n$, the form $\sum_{i=0}^n p^i F_i$ represents at least $\min\{\sum_{i=0}^n v_i, p^n\}$ different residue classes $\pmod{p^n}$, where the $x_{ij} \in \{0, 1\}$ and at least one $x_{0j} = 1$.*

3. 2-Adic Solubility

In this section, we'll show that the pair of forms (1) has nontrivial 2-adic solutions whenever $s \geq 73$. By the remarks after Lemma 3, we can prove that the system (1) has a nontrivial 2-adic solution if we can construct a primary variable at level 3. We begin with a few preliminary propositions. In these propositions, we always assume that $p = 2$ and that we are working 2-adically.

Proposition 8. *Suppose that there are two primary variables and one secondary variable at level l . Then we can create a primary variable at level (at least) $l + 1$.*

PROOF. By Lemma 5, we can contract these variables to a new variable at level (at least) $l + 1$. Since this contraction must use at least two variables, it must involve a primary variable. Thus the resulting variable is primary. \square

Proposition 9. *Suppose that there are two secondary variables at level l of different colors, and that there is also a primary variable at level l . Then we can create a primary variable at level $l + 1$.*

PROOF. If the primary variable has the same color as one of the secondary variables, then these two variables together form a zero-sum, which contracts to a primary variable at level $l + 1$. Otherwise, our set has three variables of different colors, and these three variables form a zero-sum, which contracts to a primary variable at level $l + 1$. \square

Proposition 10. *Suppose that there are three variables at level l of the same color. Then it is possible to contract two of these variables to a new variable at level exactly $l + 1$.*

PROOF. Suppose without loss of generality that the three variables have color $\binom{1}{0}$. Then the coefficient vector for each variable (ignoring the common factor of 2^l) must either be $\left\{\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right\}$ or $\left\{\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right\}$, where the “numerator” of the coefficient vector is being considered modulo 4, and the “denominator” is being considered modulo 2. Since there are three variables, there must be two with the same coefficient vectors. If we add these vectors together, we see that the sum is zero modulo 2, but nonzero modulo 4. Hence these two variables contract to a new variable at level exactly $l + 1$. \square

Proposition 11. *Suppose that there are five variables of the same color at level l . Then it is possible to construct a variable of the same color at level exactly $l + 1$. Moreover, we can do this using only two of the variables at level l .*

PROOF. Without loss of generality, suppose that the variables have color $\binom{1}{0}$. If we look at the coefficients of the variables modulo 4 (again ignoring the common factor of 2^l), there are four possibilities: $\left\{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right\}$, and $\left\{\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right\}$. Since there are five variables, two of them must have the same coefficient vector modulo 4. If these two variables are contracted, the resulting variable exists at exactly level $l + 1$, and its color will be $\binom{1}{0}$. \square

Proposition 12. *Suppose that there are three variables of the same color at level l , and suppose that a color different than these is selected. Then it is possible to use two of the variables to construct a new variable at level exactly $l + 1$ which avoids the selected color.*

PROOF. As before, assume that the variables all have color $\binom{1}{0}$ and consider their coefficient vectors modulo 4. If two of them have the same coefficient vector, then they contract to a variable of color $\binom{1}{0}$ at level $l + 1$, and we are done. Otherwise, the variables all have different coefficient vectors modulo 4, and it is not hard to see that it will always be possible to obtain variables of two different colors through contractions of two variables. One of these will avoid the selected color. \square

Now we are ready to begin the proof of our theorem when $p = 2$. We assume that the forms in (1) are 2-normalized. By Lemma 2, we have

$$\begin{aligned} m_0 &\geq 13 \\ q_0 &\geq 7 \\ m_0 + m_1 &\geq 25 \\ m_0 + q_1 &\geq 19. \end{aligned}$$

Note that by Lemma 4, along with the above inequalities, we can make at least

$$\pi_1 \geq \min \left\{ \left\lfloor \frac{m_0}{3} \right\rfloor, \left\lfloor \frac{q_0}{2} \right\rfloor \right\} \geq 3 \quad (4)$$

primary variables at level 1. We now prove that 2-adic solutions exist through a series of lemmas, which together cover all of the possible values of the q_i and m_i .

Lemma 13. *If $q_2 \neq 0$, then we can construct a primary variable at level 3.*

PROOF. As noted in (4), we know that we can construct 3 primary variables at level 1. Then by Lemma 5, we can contract them to obtain a primary variable at level 2. Since $q_2 \geq 1$, there are secondary variables of at least two different colors at level 2. Then Proposition 9 says that we can contract to a primary variable at level 3. \square

Lemma 14. *If we have $q_1 \geq 4$, then we can construct a primary variable at level 3.*

PROOF. Note that having $q_1 \geq 4$ implies that $m_1 \geq 6$. Without loss of generality, assume that the most common color at level 1 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and that the second-most common color is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then among the variables at level 1, we can find a subset having one of the following palettes:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that in any of the three possibilities, we can find three disjoint sets of two variables such that the variables in each set have different colors. To each of these sets, add one of the primary variables which can be created by (4). Then Proposition 9 allows us to create a primary variable at level 2 from each set. Since there are three variables, there exists a zero-sum among them, and this zero-sum contracts to a primary variable at level 3. \square

After the results of Lemmas 13 and 14, we may make the assumptions that $q_2 = 0$ and $q_1 \leq 3$, and we do so throughout the remainder of this section. Note that by Lemma 2, we can now assume that $m_0 \geq 16$ and $m_0 + m_1 \geq 31$.

Lemma 15. *Suppose that $q_0 \geq 10$. Then we can construct a primary variable at level 3.*

PROOF. Note that if $q_0 \geq 10$, then with $m_0 \geq 16$, Lemma 4 says that $\pi_1 \geq 5$. We now split the proof of this lemma into four cases.

Case A: $m_2 \neq 0$. After making 5 primary variables at level 1, we can use Lemma 5 twice to contract them to 2 primary variables at level 2. Since $m_2 \neq 0$, Proposition 8 allows us to contract to a primary variable at level 3.

Case B: $m_1 \geq 6$. As in the previous case, we can create two primary variables at level 2 without using any secondary variables from level 1. Now, since we have both $m_1 \geq 6$ and $q_1 \leq 3$, there must be a color at level 1 having at least three variables. By Proposition 10, we can contract two of these variables to a secondary variable at level exactly 2. Then we use Proposition 8 as above to complete the proof.

Note that by the results in these two cases, we may assume that $m_2 = 0$ and $m_1 \leq 5$, and we do so throughout the remainder of the proof. By Lemmas 2 and 4, this gives us $m_0 \geq 32$, $m_0 + m_1 \geq 37$, and $\pi_1 \geq \min\{10, \lfloor q_0/2 \rfloor\}$.

Case C: $q_0 \geq 14$. In this case, we have $\pi_1 \geq 7$. After constructing these variables, assume without loss of generality that the most common color among them is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and that the second-most common color is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If these variables have one of the palettes

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then we can find three disjoint sets of variables such that one set contains three variables of different colors, and the other two sets each contain two variables of the same color. Each of these sets contracts to a primary variable at level 2. Examining each of the other six possible palettes, one can verify that it is always possible to find three disjoint sets of variables, each containing two variables of the same color. As above, these can each be contracted to a primary variable at level 2. Hence we can construct 3 primary variables at level 2, and these can be contracted to a primary variable at level 3.

Case D: $10 \leq q_0 \leq 13$. Now we have $\pi_1 \geq 5$. Because of the bound on q_0 , we have $I_0 \geq 32 - 13 = 19$, where I_0 represents the number of variables of the majority color at level 0. As mentioned in Lemma 4, when we create the five primary variables at level 1, we use at most 10 of the variables of this color, leaving at least 9 remaining. By using Proposition 10 several times, we can contract these variables to obtain four secondary variables at level 1.

Now we consider the primary variables at level 1. As usual, we may assume that the color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ appears the most and the color $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ appears the second-most. If these five variables have one of the palettes

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{or} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then we can contract to two primary variables at level 2, with one primary variable remaining at level 1. If we have secondary variables of two different colors at level 1, then we may use Proposition 9 to create a primary variable at level 2. However, if all the secondary variables at level 1 are the same color, then we can create a secondary variable at level 2 via Proposition 10. In either case, we have three variables at level 2, at most one of which is secondary. Hence we can contract these to a variable at level 3 by Proposition 8, and this variable will be primary.

If the five primary variables at level 1 have the remaining possible palette

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then the above plan does not work, so we modify it as follows. First, using two of the variables of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we construct a primary variable at level 2. Then we have one primary variable of each possible color remaining at level 1. If the secondary variables have at least 2 different colors, then for each of these colors, we can add one secondary variable to a primary variable of the same color, and create a primary variable at level 2. This yields three primary variables at level 2. If the secondary variables all have the same color, then use Proposition 10 to create a secondary variable at level 2. Then use one of the remaining secondary variables and the primary variable of the same color to create a primary variable at level 2. Now we have two primary variables and one secondary variable at level 2, and another appeal

to Proposition 8 yields a primary variable at level 3. This completes the proof of the lemma. \square

Lemma 16. *Suppose that $8 \leq q_0 \leq 9$. Then we can construct a primary variable at level 3.*

PROOF. Note that with this bound on q_0 , Lemma 4 gives $\pi_1 \geq 4$. By studying the possible colors of these variables, we see that (after making our normal assumption about which colors are the largest), if the palette does not look like either

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then we can use these variables to make two primary variables at level 2. We now split the proof into cases.

Case A: $m_1 \geq 6$. If we can use the primary variables at level 1 to construct two primary variables at level 2, then we can finish the proof as in Lemma 15. Otherwise, begin by using two of the primary variables of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to create a primary variable at level 2. Next, since we have $m_1 \geq 6$ and $q_1 \leq 3$, there must be at least three secondary variables of the same color at level 1. By Proposition 10, we can use two of them to create a secondary variable at level exactly 2. Finally, we have two primary variables and at least 1 secondary variable remaining at level 1. Using Proposition 8, we can contract these to a primary variable at level 2. This yields two primary variables and one secondary variable at level 2, and another appeal to Proposition 8 gives us the desired primary variable.

Case B: $m_2 \neq 0$. Again, if we can use the primary variables at level 1 to construct two primary variables at level 2, then we can finish the proof as in Lemma 15. Otherwise, note that in light of Case A, we may assume that $m_1 \leq 5$. From this assumption, Lemma 2 gives us $m_0 \geq 20$, and hence $I_0 \geq 11$. At most 8 of these variables are used in creating the primary variables at level 1, leaving at least 3 remaining. By Proposition 10, we can contract two of these to a secondary variable at level 1. Now, we can contract the primary variables of the same color to a primary variable at level 2, leaving us two primary variables and one secondary variable at level 1. By Proposition 8, these variables can be contracted to a primary variable

at level 2. We now have two primary variables and one secondary variable at level 2, and once again, Proposition 8 gives us a primary variable at level 3.

Case C: $m_1 \leq 5$ and $m_2 = 0$. In this case, our conditions guarantee that we have $m_0 \geq 32$, and $I_0 \geq 32 - 9 = 23$. We need to use 8 of these in order to construct the four primary variables at level 1, leaving us with at least 15 remaining. By Proposition 10, we can use them to construct 7 secondary variables at level 1. At least three of these variables must have the same color, so we can contract two of them to a secondary variable at level 2 by Proposition 10. Next, as in the previous case, if we can use the primary variables to construct two primary variables at level 2, then we are done as above. If not, we can contract the two primary variables of the same color to a primary variable at level 2. Then we have two primary variables and a secondary variable left at level 1, and Proposition 8 yields a primary variable at level 2. As before, we now have two primary variables and one secondary variable at level 2, and another appeal to Proposition 8 completes the proof of this case. This completes the proof of this lemma. \square

Lemma 17. *Suppose that $q_0 = 7$, $1 \leq q_1 \leq 3$, and $q_2 = 0$. Then we can construct a primary variable at level 3.*

PROOF. By Lemmas 2 and 4, we have $m_0 \geq 16$, $m_0 + m_1 \geq 31$, $I_0 \geq 16 - 7 = 9$, and $\pi_1 \geq 3$. Making our usual assumption about the majority colors at level 1, the palette of primary variables at this level is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $q_1 \neq 0$, we have secondary variables of at least two different colors at level 1. It is easy to check that whichever colors they are, and whichever palette of primary variables we have, we can contract these variables to form two primary variables at level 2. Next, we point out that if either $m_2 \neq 0$ or $m_1 \geq 6$, then we can obtain a primary variable at level 3 in the same manner as in Lemma 15. Thus we may assume that $m_2 = 0$ and $m_1 \leq 5$. In that case we know from Lemma 2 that $m_0 \geq 32$ and hence $I_0 \geq 25$. We used at most 6 of the variables counted by I_0 to produce the primary variables at level 1, leaving at least 19 more. We can then use Proposition 10 to produce 9 secondary variables at level 1. Some three of these must all have the same color, and hence we can use Proposition 10 again to produce a secondary

variable at level 2. Proposition 8 now provides us with a primary variable at level 3. \square

Combining the results of the previous five lemmas, and trivially using Lemmas 2 and 4, we have the following lemma. Throughout the rest of this section, we will assume that all of the bounds in this lemma hold without explicitly stating that fact.

Lemma 18. *Suppose that (1) is a 2-normalized system. In order to prove that this system has nontrivial 2-adic solutions, we may assume that $q_0 = 7$, $q_1 = q_2 = 0$, $m_0 \geq 19$, $m_0 + m_1 \geq 31$, $m_0 + m_1 + m_2 \geq 37$, and $I_0 \geq 12$. Moreover, we can assume that $\pi_1 \geq 3$, and that to make these variables uses at most 6 of the variables counted by I_0 . Hence, after forming these primary variables, we still have a color at level 0 containing at least 6 variables.*

Lemma 19. *Suppose that we have $m_1 + m_2 \leq 1$. Then we can construct a primary variable at level 3.*

PROOF. By Lemma 18, our hypothesis implies that $m_0 \geq 36$. Assume without loss of generality that $\binom{1}{0}$ is the color at level 0 with the most elements, and divide these elements into four groups depending on whether their coefficients modulo 8 have vectors $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} * \\ 2 \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} * \\ 4 \end{smallmatrix} \right\}$, or $\left\{ \begin{smallmatrix} * \\ 6 \end{smallmatrix} \right\}$, where each asterisk can represent any odd number. Without loss of generality, assume that the coefficient vector appearing the most is $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$. Since $i_0 \geq 29$, we see by the pigeonhole principle that there are at least 8 variables with this coefficient. Moreover, since $m_0 + u(g_1) - 13 \geq t$ and there are only 7 variables not having color $\binom{1}{0}$, there must exist variables at level 0 having color $\binom{1}{0}$, but not having coefficient $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ modulo 4, and hence not $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ modulo 8 either. Our system of equations modulo 8 now looks like

$$\begin{aligned} a_1x_1^6 + \cdots + a_Mx_M^6 &+ b_1y_1^6 + \cdots + b_Ny_N^6 &+ c_1z_1^6 + \cdots + c_7z_7^6 &\equiv 0 \\ d_1y_1^6 + \cdots + d_Ny_N^6 &+ e_1z_1^6 + \cdots + e_7z_7^6 &&\equiv 0. \end{aligned} \quad (5)$$

Here, $M \geq 8$, $N \geq 1$, the variables \mathbf{x} have coefficients $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ modulo 8, the variables \mathbf{y} have coefficients $\left\{ \begin{smallmatrix} * \\ 2 \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} * \\ 4 \end{smallmatrix} \right\}$, or $\left\{ \begin{smallmatrix} * \\ 6 \end{smallmatrix} \right\}$ modulo 8, and each coefficient e_i is nonzero modulo 2. Moreover, since $M \geq 8$, Lemma 7 implies that we can solve the equation $f(\mathbf{x}) \equiv A \pmod{8}$ nontrivially for any residue A .

Now, consider the 7 variables \mathbf{z} , and add to these the variable y_1 . Since there are 8 variables, Lemma 7 allows us to nontrivially solve the equation $g(y_1, \mathbf{z}) \equiv 0 \pmod{8}$. Note that this solution must use at least one variable whose color is not $\binom{1}{0}$ modulo 2. Suppose that these variables yield $f(y_1, \mathbf{z}) \equiv -A \pmod{8}$. Then we may solve the congruence $f(\mathbf{x}) \equiv A \pmod{8}$. Then the vector $(\mathbf{x}, y_1, 0, \dots, 0, \mathbf{z})$ is a nonsingular solution of the system modulo 8, which lifts to a nontrivial solution in \mathbb{Z}_2 by Lemma 3. \square

Lemma 20. *Suppose that $m_1 + m_2 \geq 2$, that the majority color at level 0 is $\binom{1}{0}$ (this assumption being implicit in the results of Lemma 2), and that either the variables at level 1 also have color $\binom{1}{0}$ or we have $m_1 = 0$. Then we can construct a primary variable at level 3.*

PROOF. We know from Lemma 4 and the bounds in Lemma 18 that $\pi_1 \geq 3$. Our first task in this proof is to show that we can assume that at least one of these variables we can create has color $\binom{1}{0}$. Let t represent the number of variables at level 0 which have coefficients $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ modulo 4, where $*$ can represent any odd number. Since the hypotheses give $u(g_1) = 0$, Lemma 2 shows that we have $t \leq I_0 - 6$ and $t \geq 6$, where I_0 is the number of variables of color $\binom{1}{0}$ at level 0. This implies that of the variables counted by I_0 , at least 6 have coefficient $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ modulo 4, and at least 6 have coefficient $\left\{ \begin{smallmatrix} * \\ 2 \end{smallmatrix} \right\}$ modulo 4. Now, since $q_0 \geq 7$, we can find two variables (say v_1, v_2) counted by q_0 which have the same color. Adding these variables together, we see that $v_1 + v_2$ can have any of the coefficient vectors $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} 2 \\ 0 \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} 0 \\ 2 \end{smallmatrix} \right\}$, or $\left\{ \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right\}$ modulo 4. We need to show that whichever color we have, we can “complete” this to a primary variable of color $\binom{1}{0}$.

If $v_1 + v_2$ has coefficient $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\}$ modulo 4, then consider the variables with coefficient $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ modulo 4. By the pigeonhole principle, there must be two for which the asterisk represents the same number modulo 4. Adding these two variables to $v_1 + v_2$ yields a primary variable at level 1 of color $\binom{1}{0}$. If $v_1 + v_2$ has coefficient $\left\{ \begin{smallmatrix} 2 \\ 0 \end{smallmatrix} \right\}$ modulo 4, then we have two possibilities. Consider the variables of coefficient $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ modulo 4. If we have two of these where the asterisk represents different residues modulo 4, then adding these two variables to $v_1 + v_2$ yields a primary variable at level 1 of color $\binom{1}{0}$. Otherwise, we add four variables of coefficient $\left\{ \begin{smallmatrix} * \\ 0 \end{smallmatrix} \right\}$ to $v_1 + v_2$, and this will give the desired primary variable at level 1.

If $v_1 + v_2$ has coefficient $\begin{Bmatrix} 0 \\ 2 \end{Bmatrix}$ modulo 4, then we again have two possibilities. If we can find two variables at level 0 with coefficients $\begin{Bmatrix} * \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} * \\ 2 \end{Bmatrix}$ modulo 4, where the asterisks represent the same residue, then we add these variables to $v_1 + v_2$ to obtain the desired primary variable. If not, then we instead add to $v_1 + v_2$ three variables of coefficient $\begin{Bmatrix} * \\ 0 \end{Bmatrix}$ and one with coefficient $\begin{Bmatrix} * \\ 2 \end{Bmatrix}$, and we are finished (note that the asterisks in $\begin{Bmatrix} * \\ 0 \end{Bmatrix}$ all represent the same residue, and that this is a different residue than the asterisk in $\begin{Bmatrix} * \\ 2 \end{Bmatrix}$). Finally, if $v_1 + v_2$ has coefficient $\begin{Bmatrix} 2 \\ 2 \end{Bmatrix}$ modulo 4, then we have two possibilities. If there exist two variables at level 0 with coefficients $\begin{Bmatrix} * \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} * \\ 2 \end{Bmatrix}$ modulo 4, where the asterisks represent different residues, then adding these variables to $v_1 + v_2$ produces the primary variable we want. Otherwise, adding three variables with coefficient $\begin{Bmatrix} * \\ 0 \end{Bmatrix}$ and one with coefficient $\begin{Bmatrix} * \\ 2 \end{Bmatrix}$ to $v_1 + v_2$ will yield the desired variable.

Note that in order to make this variable, we use two variables from q_0 and at most four variables from I_0 . Hence after constructing this variable, we have $q'_0 = 5$ and $m'_0 \geq 13$. Then Lemma 4 gives $\pi'_1 \geq 2$, although we cannot control the colors of these variables. So we now have three primary variables at level 1, at least one of which has color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If it happens that $m_1 \geq 2$ and $m_2 \neq 0$, then we can add a primary and a secondary variable at level 1 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to produce a primary variable at level 2, leaving us with two primary variables and one secondary variable at level 1, which can be used to form a primary variable at level 2 by Proposition 8. Then Proposition 8 again allows us to construct a primary variable at level 3.

If instead we have $m_1 \geq 2$ and $m_2 = 0$, then we have two possibilities. If $m_1 \geq 4$, then we can use two secondary variables at level 1 to create a secondary variable at level 2. After doing this, we have $m'_1 \geq 2$ and $m'_2 \neq 0$, and thus we are finished by the above case. If $m_1 \leq 3$, then we must have $m_0 \geq 28$ and $I_0 \geq 21$ by Lemma 2, and after creating the three new primary variables, we have $I'_0 \geq 13$. Thus we may use Proposition 11 to create four secondary variables at level 1 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This puts us back in the first case of this paragraph, and so the desired variable can be constructed.

If we have $m_1 = 1$, then we must also have $m_2 \geq 1$. Using the ideas above, we can create an additional secondary variable at level 1. After doing this, we have $m'_1 \geq 2$ and $m'_2 \neq 0$, returning us to a case that we have

already dealt with. Finally, if we have $m_1 = 0$, then we again must have $m_2 \neq 0$. Again, the ideas above allow us to use variables at level 0 to create two secondary variables at level 1, returning us to the case where $m'_1 \geq 2$ and $m'_2 \neq 0$. Thus in either of these two final cases we can construct a primary variable at level 3, as desired. \square

Lemma 21. *Suppose that $m_1 + m_2 \geq 2$, that the majority color at level 0 is $\binom{1}{0}$ (again, this being implicit in Lemma 2), and that the variables at level 1 are not of color $\binom{1}{0}$. Then we can construct a primary variable at level 3.*

PROOF. By Lemma 20, we may assume that $m_1 \geq 1$. Without loss of generality, assume that these variables have color $\binom{0}{1}$. Using Proposition 11, we may use the variables at level 0 of color $\binom{1}{0}$ to construct four secondary variables at level 1 of color $\binom{1}{0}$. After doing this, we have $m'_0 \geq 11$ and $q'_0 = 7$. By Lemma 4, we can construct three primary variables at level 1. Hence, we now have at level 1 three primary variables, four secondary variables of color $\binom{1}{0}$ and m_1 secondary variables of color $\binom{0}{1}$.

If $m_1 \geq 3$, then we can clearly construct three primary variables at level 2, and then use these to construct a primary variable at level 3. If $m_1 = 2$, then if we have any primary variables of color $\binom{1}{0}$, then it is easy to see that we can construct three primary variables at level 2. If none of the primary variables have this color, then by Proposition 10 we can use two of the secondary variables of color $\binom{1}{0}$ to construct a secondary variable at level 2. Then at level 1 there remain two secondary variables of color $\binom{1}{0}$ and two of color $\binom{0}{1}$, and it is easy to see that we can then construct two primary variables at level 2. Thus yet another appeal to Proposition 8 allows us to construct a primary variable at level 3.

Finally, if $m_1 = 1$, note that we must have $m_2 \neq 0$. If the three primary variables all have different colors, then we can add a primary and a secondary variable of color $\binom{1}{0}$, and a primary and a secondary variable of color $\binom{0}{1}$, to produce two primary variables at level 2. Then the variables at level 2 yield a primary variable at level 3 by Proposition 8. If two of the primary variables have the same color, then adding these variables together produces a primary variable at level 2. Then we have still a primary variable and two secondary variables of different colors at level 1, and by Proposition 9, these yield a primary variable at level 2. Then we again have two primary

variables and one secondary variable at level 2, and Proposition 8 yields a primary variable at level 3. \square

The preceding lemmas show that no matter what the configuration of variables at levels 0, 1, and 2 looks like, we are always able to construct a primary variable at level 3. As mentioned in the remarks after Lemma 3, this shows that we can always find a 2-adic solution of the system. Thus the $p = 2$ case of the theorem is complete.

4. 3-Adic Solubility

Finally, in this section we will show that the pair of forms (1) has solutions in \mathbb{Q}_3 . According to the remarks following Lemma 3, we can prove that the system (1) has a nontrivial solution in \mathbb{Q}_3 if we can construct a primary variable at level 2. We first state a few preliminary propositions which will help us to accomplish this. In all of these propositions, it is assumed that $p = 3$ and that we are working 3-adically. In these propositions, if we work with the coefficient vector of a variable, then unless otherwise stated we mean the coefficients of f_l and g_l (as in Lemma 2), considered modulo 3.

Proposition 22. *Suppose that we have at level l three primary variables, and also two secondary variables which do not sum to zero (note that two variables of different colors satisfy this condition). Then we can construct a primary variable at level $l + 1$.*

PROOF. Since we have a total of 5 variables, Lemma 5 tells us that some combination of these variables add to zero. However, because the two secondary variables do not sum to zero, the zero-sum must include a primary variable. Thus the variable obtained through the construction is primary. \square

Proposition 23. *Suppose that we can find a sequence of three nonzero elements at level l which have no zero-sum subsequences. If we then add two primary variables to this sequence, then we can construct a primary variable at level $l + 1$.*

PROOF. After adding the two primary variables, we have a sequence of five variables at level l . By Lemma 5, there is a zero-sum subsequence, and

hence we can construct a variable at level at least $l + 1$. Since the original three variables have no zero-sum subsequences, the subsequence we have constructed must use at least one of the primary variables. Thus the variable we have constructed is primary. \square

Proposition 24. *Suppose that we have a sequence of four variables at the same level. If this sequence contains variables of at least three different colors, then we can find a subsequence which has three variables and has no zero-sum subsequences.*

PROOF. Suppose first that the sequence contains one variable of each color. For the first two elements of our subsequence, choose the variables of colors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Adding these variables together produces a variable of either color $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For the third element of our subsequence, choose the variable of whichever color is not the color of this sum.

Now suppose that the sequence contains only three colors. Without loss of generality, suppose that there are two elements of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and one element each of colors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If the elements of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ have the same coefficients, then they do not sum to zero. In this case, choose these two elements and either one of the other elements for the subsequence. If the two elements of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ have different coefficients, then for the first two elements of the subsequence, choose the elements of the other two colors. Then there is at most one choice of element of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which would make all three elements of the subsequence sum to zero, and so we choose the other element to complete our subsequence. \square

Proposition 25. *Suppose that we have a sequence of four elements at the same level, and that this sequence contains variables of exactly two colors. Then either there is a 3-element subsequence containing no zero-sums, or else if we add one new element to the sequence then we can make a zero-sum using that element.*

PROOF. Without loss of generality, we may suppose that the colors of the variables are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and that there are at least as many elements of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as color $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If there are three elements of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we can choose two of them having equal coefficient vectors, and these do not sum to zero. To these two elements, we add any element of color $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and obtain the

desired subsequence.

Otherwise, there are two elements of each color. If either color has two elements with the same coefficient vector, then we can use those two elements and one element of the other color for our subsequence. Otherwise, both colors have two elements with different coefficient vectors, and hence the coefficient vectors in our sequence are exactly $\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}$. Clearly, if we add any element to this sequence, then we can add to that element at most two of the original four to make a zero-sum. \square

Proposition 26. *Suppose that we have at least four secondary variables at level l , and that $q_l \geq 1$. If we can make two primary variables at level l , then we can construct a primary variable at level $l + 1$.*

PROOF. Since $q_1 \geq 1$, there are at least two colors of secondary variables at level 1. Thus, Propositions 24 and 25 say that we can either find a set of three secondary variables which have no zero-sums, or else if we add any new variable to this set, then we can make a zero-sum using that new variable. In the first case, Proposition 23 provides us with the primary variable we seek. In the second case, we simply use one of the primary variables as the “new variable”, and form our sum. \square

Proposition 27. *Suppose that we have five vectors at level zero whose coefficients have the form $\begin{Bmatrix} * \\ 9 \end{Bmatrix} \pmod{9}$, where the asterisk can represent any number that is nonzero modulo 3, and could have different values for different vectors. Then we can construct a variable having color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at level exactly 1, and this construction uses at most three of the vectors.*

PROOF. In this proof, all coefficient vectors are to be interpreted modulo 9. Without loss of generality, we may assume that the coefficient vector that appears the most is $\begin{Bmatrix} 1 \\ 9 \end{Bmatrix}$. If this vector appears at least three times, then we have

$$\begin{Bmatrix} 1 \\ 9 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 9 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 9 \end{Bmatrix} = 3 \begin{Bmatrix} 1 \\ 9 \end{Bmatrix},$$

which has color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at level 1.

Suppose now that the coefficient vector $\begin{Bmatrix} 1 \\ 9 \end{Bmatrix}$ appears exactly twice. If any of the other coefficient vectors are $\begin{Bmatrix} 2 \\ 9 \end{Bmatrix}, \begin{Bmatrix} 4 \\ 9 \end{Bmatrix},$ or $\begin{Bmatrix} 5 \\ 9 \end{Bmatrix}$, then we can easily construct the desired vector using at most three of the vectors at level 0. Since

no coefficient vector appears more than twice, both the coefficient vectors $\begin{Bmatrix} 7 \\ 9 \end{Bmatrix}$ and $\begin{Bmatrix} 8 \\ 9 \end{Bmatrix}$ must appear, and then the sum $\begin{Bmatrix} 7 \\ 9 \end{Bmatrix} + \begin{Bmatrix} 8 \\ 9 \end{Bmatrix} = 3 \begin{Bmatrix} 5 \\ 6 \end{Bmatrix}$ has the desired property.

Finally, suppose that $\begin{Bmatrix} 1 \\ 9 \end{Bmatrix}$ appears exactly once. In this case, no coefficient vector appears more than once. By the pigeonhole principle, at least one of $\begin{Bmatrix} 2 \\ 9 \end{Bmatrix}$ and $\begin{Bmatrix} 5 \\ 9 \end{Bmatrix}$ must be among our vectors. Adding this vector to $\begin{Bmatrix} 1 \\ 9 \end{Bmatrix}$ yields a vector of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at level 1. \square

Proposition 28. *Suppose that we have four vectors at level 0 of the form $\begin{Bmatrix} * \\ 9 \end{Bmatrix}$ modulo 9, and one additional vector at level 0 which is not of this form, but still has color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we can use at most three of these variables to construct a vector at level 1 which does not have color $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.*

PROOF. Again, in this proof all coefficient vectors are to be interpreted modulo 9. Without loss of generality, we may assume that the coefficient vector of the form $\begin{Bmatrix} * \\ 9 \end{Bmatrix}$ which appears the most is $\begin{Bmatrix} 1 \\ 9 \end{Bmatrix}$. Following the proof of Proposition 27 and keeping in mind that both of the sums

$$\begin{Bmatrix} 7 \\ 9 \end{Bmatrix} + \begin{Bmatrix} 8 \\ 9 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} 1 \\ 9 \end{Bmatrix} + \begin{Bmatrix} 7 \\ 9 \end{Bmatrix} + \begin{Bmatrix} 7 \\ 9 \end{Bmatrix}$$

produce variables at level 1 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we see that the only way to have four vectors of the form $\begin{Bmatrix} * \\ 9 \end{Bmatrix}$ and not have any zero-sums which yield a variable of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at level 1 is if the four vectors are $\begin{Bmatrix} 1 \\ 9 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 9 \end{Bmatrix}, \begin{Bmatrix} 8 \\ 9 \end{Bmatrix}, \begin{Bmatrix} 8 \\ 9 \end{Bmatrix}$. It is now easy to check that if we add any vector of the form $\begin{Bmatrix} * \\ 3 \end{Bmatrix}$ or $\begin{Bmatrix} * \\ 6 \end{Bmatrix}$ to these four, then it is possible to use at most three of the vectors to produce a vector at level 1 that does not have color $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. \square

Proposition 29. *Suppose that we have two variables at level 0 of the form $\begin{Bmatrix} * \\ 9 \end{Bmatrix}$ modulo 9, and also one variable of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is not of this form. Then we can construct a variable at level 1 which does not have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.*

PROOF. A variable which has color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ but does not have the form $\begin{Bmatrix} * \\ 9 \end{Bmatrix}$ must have coefficients either $\begin{Bmatrix} * \\ 3 \end{Bmatrix}$ or $\begin{Bmatrix} * \\ 6 \end{Bmatrix}$ modulo 9. Since we have three variables of the same color, we know by Lemma 6 that there exists a zero sum which includes this variable. Since the bottom coefficient of this variable is not divisible by 9, then the bottom coefficient of the zero-sum will not be divisible by 9, and hence the zero-sum will have a color (at level 1) different than $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. \square

Now we are ready to prove that the system (1) has nontrivial 3-adic solutions. As before, we assume that the forms in (1) are 3-normalized. By Lemma 2, we may now assume that

$$\begin{aligned} m_0 &\geq 13 \\ q_0 &\geq 7 \\ m_0 + m_1 &\geq 25 \\ m_0 + q_1 &\geq 19. \end{aligned}$$

Also, if we define t to be the number of variables at level 1 with coefficient $\{*_9\} \pmod{9}$, then Lemma 2 gives us (since t is an integer)

$$m_0 + u(g_1) - 13 \geq t \geq \frac{i_0}{3}.$$

By Lemma 4, we have

$$\pi_1 \geq \min \left\{ \left\lfloor \frac{m_0}{5} \right\rfloor, \left\lfloor \frac{q_0}{3} \right\rfloor \right\}.$$

We now prove that 3-adic solutions exist through a series of lemmas. In our first three lemmas, we show that all systems with $q_0 \geq 11$ have 3-adic solutions. Note that in this case the majority color at level 0 must have at least 4 variables, and so $m_0 \geq 15$.

Lemma 30. *If $q_0 \geq 11$ and either $q_1 \neq 0$ or $m_1 \geq 3$, then we can construct a primary variable at level 2.*

PROOF. If $q_1 \neq 0$, then there exist secondary variables of at least two colors at level 1. Since $m_0 \geq 15$ and $q_0 \geq 11$, Lemma 4 yields at least three primary variables at level 1. Proposition 22 now shows that we can construct a primary variable at level 2.

If $q_1 = 0$, then $m_1 \geq 3$. In this case, all of the variables at level 1 have the same color. Since there are three of them, we can choose two which have the same coefficients. These variables do not sum to zero. Since we have three primary variables, Proposition 22 now yields a primary variable at level 2. \square

Lemma 31. *If $q_0 \geq 12$, $q_1 = 0$ and $m_1 \leq 2$, then we can construct a primary variable at level 2.*

PROOF. If $1 \leq m_1 \leq 2$, then Lemma 2 gives $m_0 \geq 23$, and so Lemma 4 implies $\pi_1 \geq 4$. These primary variables, along with one secondary variable, together have a zero-sum. Clearly this zero-sum must involve a primary variable, and hence produces a primary variable.

If $m_1 = 0$, then Lemma 2 yields $m_0 \geq 25$. If it happens that $q_0 \geq 15$, then Lemma 4 gives $\pi_1 \geq 5$. These variables have a zero-sum, which is primary. Hence we may assume that $12 \leq q_0 \leq 14$, whence $i_0 \geq 11$. Now, if there are two different colors with the property that we can make a zero-sum modulo 9 using only variables of that color, then we can construct a primary variable at level 2. Hence by Lemma 7, there can be at most one color with more than 8 variables, and so $i_1, i_2, i_3 \leq 8$. Now, by Lemma 6 we can use at most three of the variables of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to make a secondary variable at level 1. Then we have $m'_0 \geq 22$ and $i'_0 \geq 8$. Since no other color has more than 8 variables, this does not change q_0 , and we must have $q'_0 \geq 12$. Then Lemma 4 guarantees that $\pi_1 \geq 4$. The five variables at level 1 have a zero-sum, which must include a primary variable. Thus the constructed variable is primary. \square

Lemma 32. *If $q_0 = 11$, $q_1 = 0$ and $m_1 \leq 2$, then we can construct a primary variable at level 2.*

PROOF. If $1 \leq m_1 \leq 2$, then $m_0 \geq 23$ and $i_0 \geq 12$. We know that the variables at level 1 have the same color. If this color is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we have $u(g_1) = 0$, and hence we know that $4 \leq t \leq m_0 - 13$. Since there are only 11 variables not of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we know that there is a variable of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which does not have coefficient $\left\{ \begin{smallmatrix} * \\ 9 \end{smallmatrix} \right\}$. Then Proposition 29 allows us to make a secondary variable at level 1 which does not have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. After constructing this variable, we have $m'_0 \geq 20, i'_0 \geq 9$. As in the previous lemma, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ must be the only color at level 0 with more than 8 variables, and hence it is still the maximal color. Thus we obtain $q'_0 = 11$ and $\pi'_1 \geq 3$. Thus we now have three primary variables at level 1, along with two secondary variables of different colors. By Proposition 22, we can construct a primary variable at level 2.

If the variables at level 1 do not have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we may assume without loss of generality that these variables have color $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $t \geq 5$, then Proposition 27 allows us to construct a secondary variable of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at

level 1. If $t = 4$, then there exist variables of color $\binom{1}{0}$ at level 0 which do not have the form $\binom{*}{9}$. In this case, Proposition 28 allows us to construct a secondary variable at level 1 which does not have color $\binom{0}{1}$. Either way, we can use at most three variables from level 0 to ensure that level 1 contains secondary variables of two different colors. After this construction, we have $m'_0 \geq 20$ and $i'_0 \geq 9$. As in the previous paragraph, we have $q'_0 = 11$ and hence $\pi'_1 \geq 3$, whence we can construct a primary variable at level 2.

Finally, if $m_1 = 0$, then we have $m_0 \geq 25$, $i_0 \geq 14$, and $5 \leq t \leq m_0 - 13$. Again, there must be at least one variable of color $\binom{1}{0}$ which does not have coefficient $\binom{*}{9}$ modulo 9. Hence we may use Propositions 27 and 29 to construct two secondary variables of different colors at level 1. Since this construction uses at most six variables, all of color $\binom{1}{0}$, we have $m'_0 \geq 19$ and $i'_0 \geq 8$. Hence $\binom{1}{0}$ is still the maximal color, and so $q'_0 = q_0 = 11$. Then Lemma 4 says that $\pi_1 \geq 3$. Combining these primary variables with the secondary variables we have constructed, an appeal to Proposition 22 gives a primary variable at level 2. \square

Lemma 33. *If $9 \leq q_0 \leq 10$ and $m_1 \geq 3$, then we can construct a primary variable at level 2.*

PROOF. If $3 \leq m_1 \leq 10$, then we have $m_0 \geq 15$, and hence Lemma 4 yields $\pi_1 \geq 3$. Since $m_1 \geq 3$, we can choose two secondary variables at level 1 which do not sum to zero. By Proposition 22, we can contract our variables to a primary variable at level 2.

If $m_1 \geq 11$ and $m_0 \geq 15$, then we again have $\pi_1 \geq 3$. Since we still have $m_1 \geq 3$, the same proof as above yields a primary variable at level 2. If $m_1 \geq 11$ and $13 \leq m_0 \leq 14$, then Lemma 4 yields two primary variables at level 1. Since $m_0 + q_1 \geq 19$, we know that $q_1 \geq 5$. Since $m_1 \geq 4$ and $q_1 \geq 1$, Proposition 26 guarantees that we can form the primary variable we seek. \square

Lemma 34. *If $9 \leq q_0 \leq 10$ and $m_1 \leq 2$, then we can construct a primary variable at level 2.*

PROOF. We treat each possible value of m_1 separately. If $m_1 = 2$, then we must have $m_0 \geq 23$, $i_0 \geq 13$, and $t \geq 5$. By Proposition 27 we can make a secondary variable at level 1 of color $\binom{1}{0}$ using at most three of the variables

from \mathbf{i}_0 . Since there are now three secondary variables at level 1, we may choose two of them which do not sum to zero. We now have $m'_0 \geq 20$ and $\mathbf{i}'_0 \geq 10$. This implies that we have $9 \leq q'_0 \leq 10$, which yields $\pi'_1 \geq 3$. These primary variables, combined with the two that we chose earlier, satisfy the conditions of Proposition 22, and so we may construct a primary variable at level 2.

If $m_1 = 1$, then $m_0 \geq 24$ and $\mathbf{i}_0 \geq 14$. This, along with Lemma 2, yields $5 \leq t \leq m_0 - 12$. Since $q_0 \leq 10$, there must be at least one variable at level 0 of color $\binom{1}{0}$ which does not have coefficient $\{*_9\}$ modulo 9. Hence by either Proposition 27 or 29, we may use at most three variables from \mathbf{i}_0 to construct a secondary variable at level 1, and can arrange so that it has a different color than the variable already there. We now have $m'_0 \geq 21$ and $\mathbf{i}'_0 \geq 11$, and so we have $9 \leq q'_0 \leq 10$ and $\pi'_1 \geq 3$. As before, Proposition 22 now gives us the primary variable we seek.

If $m_1 = 0$, then $m_0 \geq 25$ and $\mathbf{i}_0 \geq 15$. Moreover, we have $5 \leq t \leq m_0 - 13$. As above, since $q_0 \leq 10$, we have variables at level 0 of color $\binom{1}{0}$ which do not have the form $\{*_9\}$ modulo 9. By Propositions 27 and 29, we can construct two secondary variables of different colors at level 1. We now have $m'_0 \geq 19$ and $\mathbf{i}'_0 \geq 9$. If $\binom{1}{0}$ is still the largest color at level 0, then we have $q'_0 = q_0 \geq 9$. If $\binom{1}{0}$ is no longer the largest color, then having $\mathbf{i}'_0 \geq 9$ implies that we still have $q'_0 \geq 9$. In either case, Lemma 4 guarantees that we have $\pi'_1 \geq 3$. Once again, an appeal to Proposition 22 completes the proof. \square

Lemma 35. *If $7 \leq q_0 \leq 8$ and $m_1 \geq 3$, then we can construct a primary variable at level 2.*

PROOF. We divide the proof of this lemma into several cases.

Case A: $m_1 \geq 4$ and $q_1 \geq 1$. Since $m_0 \geq 13$ and $q_0 \geq 7$, we have $\pi_1 \geq 2$. Hence we are able to construct at least two primary variables at level at least 1, and Proposition 26 yields a primary variable at level 2.

Case B: $m_1 \geq 5$ and $q_1 = 0$. Since $q_1 = 0$, we must have $m_0 \geq 19$, and hence $\mathbf{i}_0 \geq 11$. There are two cases to consider. If the variables at level 1 have color $\binom{1}{0}$, then we have $u(g_1) = 0$. In this case, Lemma 4 gives $4 \leq t \leq m_0 - 13$. Since $q_0 \leq 8$, there are at least five variables of color $\binom{1}{0}$ which do not have

coefficient $\left\{ \begin{smallmatrix} * \\ 9 \end{smallmatrix} \right\}$ modulo 9. Consider one of these variables and two variables of the form $\left\{ \begin{smallmatrix} * \\ 9 \end{smallmatrix} \right\}$ modulo 9. By Proposition 29, we can construct a secondary variable at level 1 which does not have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Next, since there are at least three variables at level 1 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we may choose two of them which have the same coefficients modulo 3. These two variables, combined with the variable we just constructed, are three variables with no zero-sums. At this point, we have $m'_0 \geq 16$ and $i'_0 \geq 8$. Thus we have $q'_0 \geq 8$ and therefore $\pi'_1 \geq 2$. With two primary variables at level 1, Proposition 23 completes the proof.

Instead, suppose that the variables at level 1 have a color other than $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Without loss of generality, we may assume that this color is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Again, we may find two variables of color $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which have the same coefficients. If $t \geq 5$, then by Proposition 27 we can use three variables with coefficients $\left\{ \begin{smallmatrix} * \\ 9 \end{smallmatrix} \right\}$ modulo 9 to create a secondary variable at level 2 with color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $t = 4$, then since $i_0 \geq 11$ we have a variable at level 0 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which does not have the form $\left\{ \begin{smallmatrix} * \\ 9 \end{smallmatrix} \right\}$. Then by Proposition 28 we can construct a variable at level 1 of color other than $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Either way, we obtain a set of three secondary variables at level 1 with no zero-sums. After constructing this secondary variable, we have $m'_0 \geq 16$ and $i'_0 \geq 8$. This again implies that $q'_0 = q_0$, and hence that $\pi'_1 \geq 2$. Thus we may construct two primary variables at level 1, and Proposition 23 completes the proof of this case.

Case C: $3 \leq m_1 \leq 4$ and $q_1 = 0$. As before, we can find two variables at level 1 which have equal coefficients. Moreover, by Lemmas 2 and 4, we have $m_0 \geq 21$, $i_0 \geq 13$, and $m_0 + u(g_1) - 13 \geq t \geq 5$. We again split the proof into cases based on the color of the variables at level 1. If these variables have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $t \leq m_0 - 13$. Therefore there are at least five variables at level 0 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which do not have coefficient $\left\{ \begin{smallmatrix} * \\ 9 \end{smallmatrix} \right\}$ modulo 9. Since we also have at least two variables at level 0 which do have this form, Proposition 29 allows us to construct a secondary variable at level 1 which does not have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We now have three secondary variables at level 1 which have no zero-sums. Moreover, we now have $m'_0 \geq 18$, $i'_0 \geq 10$, and $q'_0 = q_0 \geq 7$, and hence $\pi'_1 \geq 2$. After we construct two primary variables at level 1, Proposition 23 yields a primary variable at level 2.

On the other hand, if the variables at level 1 have a different color, then

without loss of generality we may assume that this color is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Again, we may choose two of these variables which have equal coefficients modulo 9. Since $t \geq 5$, Proposition 27 allows us to construct a secondary variable of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at level 1. Then the three secondary variables at level 1 have no zero-sums. After this construction, we still have $\pi'_1 \geq 2$ as before, and so Proposition 23 again yields the primary variable we seek.

Case D: $m_1 = 3$ and $q_1 \geq 1$. Suppose first that there exist exactly two colors of variables at level 1. If there is a color which has two variables with equal coefficients modulo 3, then we are finished as in previous cases. So suppose that the two variables of the same color have different coefficients. We have $m_0 \geq 22$ and $i_0 \geq 14$, and $t \geq 5$. By Proposition 27, we can construct a variable at level 1 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If the secondary variables of the same color also have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we can now choose two secondary variables of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which do not sum to zero, and one secondary variable of a different color to obtain a trio of secondary variables with no zero-sums. As before, we still have $\pi'_1 \geq 2$, and so Proposition 23 gives us the primary variable we want.

If the third variable at level 1 has color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then our construction yields another secondary variable of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If these variables have equal coefficients, then we are finished as in previous cases. If they have different coefficients, then any three of the variables at level 1 have a zero-sum. Hence, Proposition 25 tells us that if we can make a single primary variable at level 1, then we can use that variable in a zero-sum to produce a primary variable at level at least 2. Since we still have $\pi'_1 \geq 2$, we are finished in this case. Finally, suppose that none of the variables at level 1 have color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then after our construction, we have secondary variables of three different colors at level 1 and $\pi'_1 \geq 2$. Then Propositions 24 and 23 provide a primary variable at level 2.

There remains the possibility that the variables counted by m_1 represent three different colors. In this case, since $t \geq 5$, we can make a secondary variable at level 1 of color $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. After doing this we will have four secondary variables at level 1, and hence by Proposition 24, we can find three of them which have no zero-sums. After creating this variable, we still have $\pi'_1 \geq 2$, and so Proposition 23 provides a primary variable at level 2. This completes the proof of the lemma. \square

Lemma 36. *If $7 \leq q_0 \leq 8$ and $m_1 \leq 2$, then we can construct a primary variable at level 2.*

PROOF. In this proof, we treat each possible value of m_1 separately. If $m_1 = 2$, then we have $m_0 \geq 23$, $i_0 \geq 15$, and $5 \leq t \leq m_0 - 11$. Using Propositions 27 and 29, we can create two secondary variables at level 1 of different colors. After creating these variables, we have $m'_1 \geq 4$, $q'_1 \geq 1$ and $\pi'_1 \geq 2$. Then Proposition 26 allows us to create the primary variable we want.

If $m_1 = 1$, then $m_0 \geq 24$, $i_0 \geq 16$, and $6 \leq t \leq m_0 - 12$. As above, we can construct two secondary variables at level 1 of different colors. After doing this, we still have at least 10 variables at level 0 which have color $\binom{1}{0}$, and by Lemma 6, we can use at most three of these to construct another secondary variable at level 1. This gives us a total of four secondary variables at level 1. After these constructions, we have $q'_1 \geq 1$, $m'_0 \geq 15$, $i'_0 \geq 7$, and $q'_0 \geq 7$ (note that we might have $q_0 = 8$ and $q'_0 = 7$), and hence $\pi'_1 \geq 2$. Then Proposition 26 allows us to construct a primary variable at level 2.

Finally, if $m_1 = 0$, then $m_0 \geq 25$, $i_0 \geq 17$, and $6 \leq t \leq m_0 - 13$. As in previous cases we construct two secondary variables at level 1, one of color $\binom{1}{0}$ and one not of this color. After this, we still have at least 11 variables at level 0 of color $\binom{1}{0}$, and hence by Lemma 6 we can use these to create three more secondary variables at level 1. Of the five secondary variables we have constructed, we wish to select 3 of them which have no zero-sums. If the five variables represent at least three colors, then Proposition 24 allows us to do this. If the variables represent exactly two colors, then some three of them have the same color, and two of these have the same coefficients. These two variables, along with one variable of the other color, are a collection of three variables with no zero-sums.

We now deconstruct the two variables that we have not selected, returning their component parts to level 0. After this, we have constructed a total of three variables at level 1, using at most 9 variables from level 0, all of color $\binom{1}{0}$. Hence we have $m'_0 \geq 16$, $i'_0 \geq 8$, $q'_0 = q_0$, and hence $\pi'_1 \geq 2$. Thus Proposition 23 allows us to construct a primary variable at level 2. Since this is the final case, this completes the proof of the lemma. \square

Finally, we point out that since we know from normalization that $q_0 \geq 7$, these lemmas complete the proof of the theorem.

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